Radiant Interchange in a Nonisothermal Rectangular Cavity

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The influence of an exponentially decreasing temperature distribution on radiant interchange in a rectangular cavity is investigated using Ambarzumian's method. The rectangular cavity is defined by two semi-infinite parallel, diffuse gray surfaces, and a finite black surface. The radiosity at the edge of the cavity is shown to satisfy a nonlinear integral equation which is easily solved by iteration. For an isothermal cavity, the radiosity at the edge of the cavity is equal to the inverse of the square root of the emittance. Without calculating the radiosity distribution inside the cavity, the over-all heat transfer is expressed in terms of the radiosity at the edge of the cavity. Numerical results are presented for a wide range of temperature distributions and reflectances.

Nomenclature

a_i, b_i = constants in generalized emissive power distribution
$B(x, m)$ = dimensionless radiosity for exponentially decay temperature
$B^*(x, m)$ = radiosity for exponentially decay temperature
$\tilde{B}(\tilde{x})$ = radiosity, radiative flux leaving the surface
$\bar{B}(s, m)$ = Laplace transform of $B(x, m)$
$\tilde{G}(\tilde{x})$ = irradiation, radiative flux incident on the surface
h = spacing between plates
h_n = $\dfrac{\rho}{2}\displaystyle\int_0^\infty t^n J_1(t) H(t)\, dt$
$H(m)$ = radiosity at the edge of the cavity, $B(0, m)$
$J_1(t)$ = Bessel function of order one
$K(|x-y|)$ = kernel of the integral equation
m = temperature distribution parameter
$q(x, m)$ = dimensionless local heat flux
$q^*(x, m)$ = local heat flux
$Q(m)$ = dimensionless over-all heat transfer
Q^* = over-all heat transfer
Q_o = dimensionless over-all heat transfer when the temperature of surfaces one and two are zero
T_b = temperature of surface 3
T_o = reference temperature
T_w = cavity's wall temperature
x, y = $\tilde{x}/h$, $\tilde{y}/h$
$\tilde{x}, \tilde{y}$ = distance into the cavity
α_o = zeroth moment of the H-function
α_1 = first moment of the H-function
ΔT = maximum temperature difference in cavity, see Eq. (77)
ε = emittance
ρ = reflectance, $1-\varepsilon$
σ = Stefan-Boltzmann constant
$\phi(x)$ = dimensionless radiosity for the case when the temperatures of surfaces one and two are zero
$\Phi(x)$ = dimensionless radiosity for an emissive power variation like the kernel, see Eq. (21)

Introduction

THIS paper is concerned with the influence of a nonisothermal temperature distribution on the radiant interchange in a rectangular cavity. The analysis is carried out for a wall temperature distribution which decreases exponentially with depth into the cavity. The cavity walls are taken to be gray, diffuse emitters and reflectors of radiant energy. The mathematical formulation of this problem results in a linear Fredholm integral equation for the radiosity which is solved by Ambarzumian's method.[1,2] This powerful technique was developed to study radiative transfer in a scattering medium. The radiosity at the cavity's edge and the over-all heat transfer from the cavity are determined without knowing the radiosity distribution within the cavity.

Most investigations[3-6] of radiant interchange in a cavity assume isothermal walls; however, a few[6-9] have considered nonisothermal walls. For a finite cylindrical cavity, Sparrow[7] analytically studied a linear temperature distribution with an exponential kernel approximation, while Peary[8] numerically analyzed a polynomial temperature distribution with successive approximations. Since the kernels for the circular-arc cavity and the spherical cavity are separable, closed-form solutions to these geometries can be obtained for an arbitrary temperature distribution.[6] Crosbie and Look[9] found a closed-form solution for two infinite parallel plates with cosine varying temperature distributions.

Recently, Crosbie and Sawheny[10] applied Ambarzumian's method for the first time to a radiant interchange problem: a rectangular cavity subjected to an exponentially decreasing heat flux. The present investigation demonstrates another powerful aspect of Ambarzumian's method, i.e., ability to predict the over-all heat transfer without knowing the radiosity distribution.

Physical Model

The rectangular cavity is formed by two semi-infinite parallel, diffuse, gray surfaces (1 and 2) and a finite black isothermal surface 3. Surface 3 could be an apparent surface representing the radiation from the surroundings at temperature T_b. This geometry is illustrated in Fig. 1. The cavity is filled with a non-participating medium. The temperature distribution and radiative properties of surfaces one and two are assumed to be identical. Because of the symmetry of the enclosure only one surface is considered in the analysis.

The radiosity of surface one can be expressed as

$$\tilde{B}(\tilde{x}) = \varepsilon\sigma T_w^4(\tilde{x}) + \rho\tilde{G}(\tilde{x}) \tag{1}$$

where ε is the emittance, ρ is the reflectance, σ is the Stefan-Boltzmann constant, and T is temperature. The first term on the right side of Eq. (1) represents the radiant flux emitted, and $\tilde{G}(\tilde{x})$ is the irradiation from the other surfaces. Expressing the

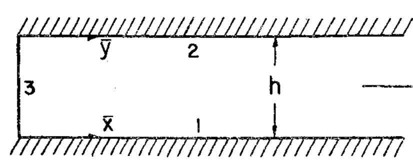

Fig. 1 Rectangular cavity.

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irradiation in terms of the radiosity yields the following linear integral equation:

$$\tilde{B}(\bar{x}) = \varepsilon\sigma T_w^4(\bar{x}) + \frac{\rho}{2}\sigma T_b^4\left[1 - \frac{\bar{x}}{(\bar{x}^2+h^2)^{1/2}}\right] + \frac{\rho}{2}\int_0^\infty \frac{h^2\tilde{B}(\bar{y})\,d\bar{y}}{[(\bar{x}-\bar{y})^2+h^2]^{3/2}} \tag{2}$$

The second term accounts for incident radiation from surface 3, while the third accounts for incident radiation from surface 2. Introducing $x = \bar{x}/h$ and $y = \bar{y}/h$, and assuming the temperature decays exponentially, i.e.,

$$T_w(x) = T_o \exp(-mx/4) \tag{3}$$

the integral equation for the radiosity (2) takes the form

$$B^*(x,m) = \varepsilon\sigma T_o^4 e^{-mx} + \frac{\rho}{2}\sigma T_b^4\left[1 - \frac{x}{(x^2+1)^{1/2}}\right] + \frac{\rho}{2}\int_0^\infty B^*(y,m)K(|x-y|)\,dy \tag{4}$$

where the kernel is defined as

$$K(|x-y|) = K(x,y) = 1/[(x-y)^2+1]^{3/2} \tag{5}$$

Considering the semi-infinite geometry, the exponential variation seems a natural choice to represent the temperature distribution along the cavity's wall. Since integral Eq. (4) is linear, $B^*(x,m)$ can be expressed as the superposition of two functions

$$B^*(x,m) = \varepsilon\sigma T_o^4 B(x,m) + \rho\sigma T_b^4\phi(x) \tag{6}$$

where $B(x,m)$ and $\phi(x)$ satisfy the following two integral equations

$$B(x,m) = e^{-mx} + \frac{\rho}{2}\int_0^\infty B(y,m)K(|x-y|)\,dy \tag{7}$$

and

$$\phi(x) = \frac{1}{2}\left[1 - \frac{x}{(x^2+1)^{1/2}}\right] + \frac{\rho}{2}\int_0^\infty \phi(y)K(|x-y|)\,dy \tag{8}$$

Physically, $B(x,m)$ is the dimensionless radiosity for the case when the temperature of black surface three is zero. And $\phi(x)$ represents the dimensionless radiosity ($B^*/\rho\sigma T_b^4$) for the case when the temperatures of surfaces one and two are zero.

The local heat flux can be written as

$$q^*(x,m) = (\varepsilon/\rho)[\sigma T_o^4 e^{-mx} - B^*(x,m)] \tag{9}$$

Using Eq. (6), the local heat flux can be expressed as

$$q^*(x,m) = \varepsilon\sigma T_o^4 q(x,m) - \varepsilon\sigma T_b^4\phi(x) \tag{10}$$

where

$$q(x,m) = (1/\rho)[e^{-mx} - \varepsilon B(x,m)] \tag{11}$$

The over-all heat transfer per unit length of cavity is

$$Q^* = 2h\int_0^\infty q^*(x,m)\,dx \tag{12}$$

or

$$Q^* = 2h\varepsilon\sigma T_o^4 Q(m) - 2h\varepsilon\sigma T_b^4 Q_o \tag{13}$$

where

$$Q(m) = \int_0^\infty q(x,m)\,dx = \frac{1}{\rho}\int_0^\infty [e^{-mx} - \varepsilon B(x,m)]\,dx \tag{14}$$

and

$$Q_o = \int_0^\infty \phi(x)\,dx \tag{15}$$

$Q(m)$ is the heat transfer from the cavity when the temperature of surface three is zero. $Q(m)$ depends on the temperature distribution parameter m and the reflectance ρ. Q_o is the heat transfer to the cavity from the black surface three when the temperature of surfaces one and two is zero.

A standard approach in solving this problem would be as follows: a) solve linear integral Eqs. (7) and (8) numerically by successive approximations for the spatial radiosity distribution; b) calculate the local heat flux distribution, Eq. (11); and c) determine the over-all heat transfer, Eq. (14), by integrating the local heat flux. This procedure would be carried out for each value of m. Ambarzumian's method goes against this philosophy and considers both x and m as variables. Ambarzumian's method for a semi-infinite geometry is as follows: a) Eq. (7) is transformed into a integro-differential equation of the initial-value type, b) a nonlinear integral equation for radiosity at the cavity's edge is found and solved by iteration, and c) the over-all heat transfer is related to the radiosity at the cavity's edge. Thus, Ambarzumian's method yields the over-all heat transfer for all m values by solving an integral equation only once.

Dimensionless Radiosity, $B(x,m)$

In this section, Ambarzumian's method[1,2] is applied to integral Eq. (7). The analysis follows that of Crosbie and Sawheny.[10]

Integro-Differential Equation

The Eq. (7) for dimensionless radiosity $B(x,m)$ can be rewritten as follows

$$B(x,m) = e^{-mx} + \frac{\rho}{2}\int_0^x B(y,m)K(x-y)\,dy + \frac{\rho}{2}\int_x^\infty B(y,m)K(y-x)\,dy \tag{16}$$

Substituting $z = x-y$ in the first integral and $z = y-x$ in the second integral of the right-hand side of Eq. (16) yields

$$B(x,m) = e^{-mx} + \frac{\rho}{2}\int_0^x B(x-z,m)K(z)\,dz + \frac{\rho}{2}\int_0^\infty B(x+z,m)K(z)\,dz \tag{17}$$

Differentiating Eq. (17) with respect to x gives

$$\frac{\partial B(x,m)}{\partial x} = -me^{-mx} + \frac{\rho}{2}B(0,m)K(x) + \frac{\rho}{2}\int_0^x \frac{\partial B(x-z,m)}{\partial x}K(z)\,dz + \frac{\rho}{2}\int_0^\infty \frac{\partial B(x+z,m)}{\partial x}K(z)\,dz \tag{18}$$

Now substituting $y = x-z$ in the first integral and $y = x+z$ in the second integral of the right-hand side of Eq. (18) yields the following integral equation for the derivative of $B(x,m)$:

$$\frac{\partial B(x,m)}{\partial x} = -me^{-mx} + \frac{\rho}{2}B(0,m)K(x) + \frac{\rho}{2}\int_0^\infty \frac{\partial B(y,m)}{\partial y}K(|x-y|)\,dy \tag{19}$$

The solution of Eq. (19) is determined by the superpositions of solutions. Multiply Eq. (7) by $mJ_1(m)dm/2$ and integrating from 0 to ∞ yields

$$\Phi(x) = \tfrac{1}{2}K(x) + \frac{\rho}{2}\int_0^\infty K(|x-y|)\Phi(y)\,dy \tag{20}$$

where

$$\Phi(x) = \frac{1}{2}\int_0^\infty mB(x,m)J_1(m)\,dm \tag{21}$$

In obtaining Eq. (20), the following identity was employed:

$$K(x) = \int_0^\infty te^{-xt}J_1(t)\,dt = 1/(x^2+1)^{3/2} \tag{22}$$

Superimposing Eqs. (7) and (20) yields the integro-differential equation for $B(x,m)$

$$\frac{\partial B(x,m)}{\partial x} = -mB(x,m) + \frac{\rho}{2}B(0,m)\int_0^\infty tB(x,t)J_1(t)\,dt \tag{23}$$

H-Function

In the physical sense, H-function is the radiosity at the edge of the cavity, $B(0,m)$. The aim of this section is to achieve a nonlinear integral equation for H-function which is convenient for numerical solution. Evaluating Eq. (7) at $x=0$ and employing identity (22) yields

$$B(0,m) = 1+\frac{\rho}{2}\int_0^\infty B(y,m)\int_0^\infty te^{-yt}J_1(t)\,dt\,dy \qquad (24)$$

By changing the order of integration, Eq. (24) can be written as

$$B(0,m) = 1+\frac{\rho}{2}\int_0^\infty tJ_1(t)\bar{B}(t,m)\,dt \qquad (25)$$

where $\bar{B}(t,m)$ is the Laplace transform of $B(y,m)$, $\bar{B}(t,m) = \int_0^\infty e^{-ty}B(y,m)\,dy$. $\bar{B}(t,m)$ can be expressed in terms of $B(0,m)$ by applying the Laplace transform to Eq. (23), i.e.,

$$(s+m)\bar{B}(s,m) = B(0,m)\left[1+\frac{\rho}{2}\int_0^\infty t\bar{B}(s,t)J_1(t)\,dt\right] \qquad (26)$$

The Laplace transform of $B(x,m)$ can be shown to be symmetric, $\bar{B}(s,m) = \bar{B}(m,s)$. Thus, Eq. (26) becomes

$$\bar{B}(s,m) = B(0,m)B(0,s)/(s+m) \qquad (27)$$

Substitution of Eq. (27) into Eq. (25) yields

$$B(0,m) = 1+\frac{\rho}{2}B(0,m)\int_0^\infty \frac{tJ_1(t)}{m+t}B(0,t)\,dt \qquad (28)$$

Utilizing the definition of the H-function, i.e., $H(m) = B(0,m)$, Eq. (28) becomes

$$H(m) = 1+\frac{\rho}{2}H(m)\int_0^\infty \frac{tJ_1(t)}{m+t}H(t)\,dt \qquad (29)$$

Since Eq. (29) would converge slowly on application of an iterative technique, an alternate form is sought. Using the transform $t/(m+t) = 1-m/(m+t)$ and dividing Eq. (29) by $H(m)$ yields

$$\frac{1}{H(m)} = 1-\frac{\rho}{2}\alpha_o+\frac{\rho}{2}\int_0^\infty \frac{mJ_1(t)}{m+t}H(t)\,dt \qquad (30)$$

where α_o is the zeroth moment of the H-function, i.e.,

$$\alpha_o = \int_0^\infty H(m)J_1(m)\,dm \qquad (31)$$

The zeroth moment can be determined without calculating the H-function. Multiplying Eq. (29) by $J_1(m)\,dm$ and integrating over the interval $(0,\infty)$ gives

$$\alpha_o = 1+\frac{\rho}{2}\int_0^\infty\int_0^\infty \frac{tH(m)J_1(m)H(t)J_1(t)}{m+t}\,dt\,dm \qquad (32)$$

Interchanging the dummy variables m and t and adding the equation to Eq. (32) yields a quadratic equation of the form $\rho\alpha_o^2-4\alpha_o+4=0$ with solution

$$\alpha_o = \frac{2}{\rho}[1-(1-\rho)^{1/2}] = \frac{2}{\rho}[1-(\varepsilon)^{1/2}] = \frac{2}{1+(\varepsilon)^{1/2}} \qquad (33)$$

Thus, Eq. (30) becomes

$$\frac{1}{H(m)} = (1-\rho)^{1/2}+\frac{\rho}{2}\int_0^\infty \frac{mJ_1(t)}{m+t}H(t)\,dt \qquad (34)$$

Setting $m=0$ in Eq. (34) yields a closed-form solution for the isothermal cavity, i.e.,

$$H(0) = B(0,0) = 1/(1-\rho)^{1/2} = 1/(\varepsilon)^{1/2} \qquad (35)$$

Dimensionless Radiosity, $\phi(x)$

Equation (8) for the dimensionless radiosity, $\phi(x)$, falls into the category of a linear Fredholm integral equation of the second kind. $\phi(x)$ depends upon the depth in the cavity x and the reflectance ρ. An expression for $\phi(x)$ in terms of the dimensionless radiosity $B(x,m)$ along with the development of a differential equation for $\phi(x)$ is presented in this section.

Multiplying Eq. (7) by $J_1(m)dm/2$ and integrating from 0 to ∞ yields

$$\frac{1}{2}\int_0^\infty J_1(m)B(x,m)\,dm = \frac{1}{2}\left[1-\frac{x}{(x^2+1)^{1/2}}\right]+$$
$$\frac{\rho}{2}\int_0^\infty\left[\frac{1}{2}\int_0^\infty J_1(m)B(y,m)\,dm\right]K(|x-y|)\,dy \qquad (36)$$

with the aid of identity

$$\int_0^\infty e^{-xm}J_1(m)\,dm = 1-[x/(x^2+1)^{1/2}] \qquad (37)$$

Comparing Eqs. (8) and (36) reveals that

$$\phi(x) = \frac{1}{2}\int_0^\infty J_1(m)B(x,m)\,dm \qquad (38)$$

Since a differential equation is easier to solve than an integral equation, Eq. (8) is transformed into a differential equation for $\phi(x)$. Multiplying Eq. (23) by $J_1(m)\,dm$ and integrating over the interval $(0,\infty)$ yields

$$\int_0^\infty J_1(m)\frac{\partial B(x,m)}{\partial x}\,dm = -\int_0^\infty mJ_1(m)B(x,m)\,dm+$$
$$\frac{\rho}{2}\int_0^\infty J_1(m)B(0,m)\int_0^\infty nB(x,n)J_1(n)\,dn\,dm \qquad (39)$$

or

$$\frac{\partial}{\partial x}\int_0^\infty J_1(m)B(x,m)\,dm = -\int_0^\infty mJ_1(m)B(x,m)\,dm+$$
$$\frac{\rho}{2}\int_0^\infty J_1(m)H(m)\,dm\int_0^\infty nB(x,n)J_1(n)\,dn \qquad (40)$$

Using Eqs. (21) and (38), Eq. (40) can be written as

$$\frac{\partial\phi(x)}{\partial x} = -\Phi(x)+\frac{\rho}{2}\alpha_o\Phi(x) = -(\varepsilon)^{1/2}\Phi(x) \qquad (41)$$

The initial condition for Eq. (41) is

$$\phi(0) = \tfrac{1}{2}\alpha_o = (1/\rho)[1-(\varepsilon)^{1/2}] = 1/[1+(\varepsilon)^{1/2}] \qquad (42)$$

Comparison of the integro-differential Eq. (41) for $\phi(x)$ with the Eq. (23) for $B(x,0)$ reveals that

$$\phi(x) = a-(\varepsilon/\rho)B(x,0) \qquad (43)$$

The constant a is found equal to $1/\rho$ by using conditions (35) and (42), and the relationship between $\phi(x)$ and $B(x,0)$ becomes

$$\phi(x) = (1/\rho)-(\varepsilon/\rho)B(x,0) \qquad (44)$$

Setting $\varepsilon = 0$ in equation yields a closed-form solution of unity for $\phi(x)$.

Dimensionless Heat Transfer, $Q(m)$ and Q_o

Physically $Q(m)$ is the over-all dimensionless heat transfer from the cavity for the case when the temperature of the black surface three is zero. Referring back to the definition of $Q(m)$, Eq. (14), the over-all heat transfer can be expressed in terms of the Laplace transform of the radiosity, i.e.,

$$Q(m) = \frac{1}{\rho m}-\frac{\varepsilon}{\rho}\int_0^\infty B(x,m)\,dx = \frac{1}{\rho m}-\frac{\varepsilon}{\rho}\bar{B}(0,m) \qquad (45)$$

Substitution of Eq. (27) into Eq. (45) yields

$$Q(m) = [1-(\varepsilon)^{1/2}H(m)]/\rho m \qquad (46)$$

An alternate form is obtained using Eq. (31) and is given by

$$Q(m) = \frac{H(m)}{2}\int_0^\infty \frac{J_1(t)H(t)}{m+t}\,dt \qquad (47)$$

When surfaces one and two of the cavity are isothermal ($m=0$), the over-all heat transfer can be expressed from Eq. (47) as

$$Q(0) = \alpha_1/2(\varepsilon)^{1/2} \qquad (48)$$

where α_1 is the first moment of the H-function

$$\alpha_1 = \int_0^\infty J_1(t)H(t)(dt/t) \qquad (49)$$

Q_o represents the over-all dimensionless heat transfer from the cavity when the temperatures of surfaces one and two are zero. Using expression (38) for $\phi(x)$, Eq. (15) gives

$$Q_o = \frac{1}{2}\int_0^\infty \int_0^\infty J_1(m)B(x,m)\,dm\,dx \tag{50}$$

Interchanging the order of integration yields

$$Q_o = \frac{1}{2}\int_0^\infty J_1(m)\bar{B}(0,m)\,dm \tag{51}$$

and using Eq. (27) into Eq. (51) gives

$$Q_o = \frac{1}{2}\int_0^\infty \frac{J_1(m)H(m)}{(\varepsilon)^{1/2}m}\,dm = \frac{\alpha_1}{2(\varepsilon)^{1/2}} \tag{52}$$

This result is identical to the dimensionless heat transfer for an isothermal cavity $Q(0)$. Physically this must be the case, since the over-all heat transfer Q^* is zero when $T_o = T_b$.

Asymptotic Expansions

When m is small, the following relation

$$m/(t+m) = (m/t) - [m^2/t(t+m)] \tag{53}$$

is utilized in rewriting Eq. (34) as

$$\frac{1}{H(m)} = (\varepsilon)^{1/2} + \frac{\rho}{2}\alpha_1 m - \frac{\rho}{2}\int_0^\infty \frac{m^2 J_1(t)}{t(t+m)}H(t)\,dt \tag{54}$$

or

$$H(m) = 2/[2(\varepsilon)^{1/2} + \rho\alpha_1 m - \rho m^2 I(m)] \tag{55}$$

where

$$I(m) = \int_0^\infty \frac{J_1(t)H(t)}{t(t+m)}\,dt$$

Substitution of Eq. (55) into Eq. (46) gives

$$Q(m) = [\alpha_1 - mI(m)]/[2(\varepsilon)^{1/2} + \rho\alpha_1 m - \rho m^2 I(m)] \tag{56}$$

Neglecting the integral term $m^2 I(m)$ in Eq. (55) yields

$$H(m) \simeq \frac{2}{2(\varepsilon)^{1/2} + \rho\alpha_1 m} \tag{57}$$

However, the term $mI(m)$ in Eq. (56) cannot be neglected in comparison to $\rho\alpha_1 m$.

When m is large, the following expansion

$$\frac{t}{m+t} = \frac{t}{m} - \frac{t^2}{m^2} + \frac{t^3}{m^3} - \frac{t^4}{m^4} + \cdots \tag{58}$$

is used in rewriting Eq. (29) as

$$\frac{1}{H(m)} \sim 1 - \frac{h_1}{m} + \frac{h_2}{m^2} - \frac{h_3}{m^3} + \frac{h_4}{m^4} - \cdots \tag{59}$$

where

$$h_n = \frac{\rho}{2}\int_0^\infty t^n J_1(t)H(t)\,dt \tag{60}$$

Inversion of this series yields

$$H(m) \sim 1 + \frac{h_1}{m} + \frac{(h_1^2 - h_2)}{m^2} + \frac{(h_1^3 + h_3 - 2h_1h_2)}{m^3} + \frac{(2h_1h_3 - 3h_1^2h_2 - h_4 + h_2^2 + h_1^4)}{m^4} + \cdots \tag{61}$$

However, h_1 and h_2 are related in a simple fashion. Rearrangement of Eq. (34) gives

$$(\varepsilon)^{1/2}H(m) = 1 - \frac{\rho}{2}H(m)\int_0^\infty \frac{m}{m+t}J_1(t)H(t)\,dt \tag{62}$$

Multiplication of this equation by $(\rho/2)m^2 J_1(m)\,dm$ and integration from 0 to ∞ yields

$$(\varepsilon)^{1/2}h_2 = -\frac{\rho^2}{4}\int_0^\infty \int_0^\infty \frac{m^3}{m+t}J_1(m)J_1(t)H(t)H(m)\,dt\,dm \tag{63}$$

upon utilization of the following identity

$$\int_0^\infty m^2 J_1(m)\,dm = 0 \tag{64}$$

Interchange of dummy variables m and t and addition of the resulting equation to Eq. (63) yields

$$(\varepsilon)^{1/2}h_2 = -\frac{\rho^2}{8}\int_0^\infty \int_0^\infty \frac{m^3+t^3}{m+t}J_1(m)J_1(t)H(m)H(t)\,dt\,dm \tag{65}$$

or

$$(\varepsilon)^{1/2}h_2 = -\frac{\rho^2}{8}\int_0^\infty \int_0^\infty (m^2 - mt + t^2)J_1(m)J_1(t)H(m)H(t)\,dt\,dm \tag{66}$$

The right-hand side of Eq. (66) can be now integrated, i.e.,

$$2(\varepsilon)^{1/2}h_2 = h_1^2 - \rho\alpha_o h_2 \tag{67}$$

and with Eq. (33) yields

$$2h_2 = h_1^2 \tag{68}$$

A similar relation can be obtained by multiplying Eq. (62) by $(\rho/2)m^4 J_1(m)\,dm$ and integrating from 0 to ∞, i.e.,

$$(\varepsilon)^{1/2}h_4 = -\frac{\rho^2}{8}\int_0^\infty \int_0^\infty \frac{m^5}{m+t}J_1(m)J_1(t)H(m)H(t)\,dt\,dm \tag{69}$$

upon utilization of the following identity

$$\int_0^\infty m^4 J_1(m)\,dm = 0 \tag{70}$$

Interchange of dummy variables m and t and addition of the resulting equation to Eq. (69) yields

$$(\varepsilon)^{1/2}h_4 = -\frac{\rho^2}{8}\int_0^\infty \int_0^\infty \frac{m^5+t^5}{m+t}J_1(m)J_1(t)H(m)H(t)\,dt\,dm \tag{71}$$

or

$$(\varepsilon)^{1/2}h_4 = -\frac{\rho^2}{8}\int_0^\infty \int_0^\infty (m^4 - m^3t + m^2t^2 - mt^3 + t^4) \times J_1(m)J_1(t)H(m)H(t)\,dt\,dm \tag{72}$$

The right-hand side of Eq. (72) can be integrated, i.e.,

$$2(\varepsilon)^{1/2}h_4 = 2h_3h_1 - h_2^2 - \rho\alpha_o h_4 \tag{73}$$

and with Eqs. (33) and (68) yields

$$h_4 = h_3h_1 - \tfrac{1}{8}h_1^4 \tag{74}$$

Thus with Eqs. (68) and (74), Eq. (61) becomes

$$H(m) \sim 1 + \frac{h_1}{m} + \frac{h_2}{m^2} + \frac{h_3}{m^3} + \frac{h_4}{m^4} + \cdots \tag{75}$$

Substitution of Eq. (75) into Eq. (46) gives

$$Q(m) \sim \frac{1}{[1+(\varepsilon)^{1/2}]m} - \frac{(\varepsilon)^{1/2}h_1}{\rho m^2} - \frac{(\varepsilon)^{1/2}h_2}{\rho m^3} - \frac{(\varepsilon)^{1/2}h_3}{\rho m^4} - \frac{(\varepsilon)^{1/2}h_4}{\rho m^5} - \cdots \tag{76}$$

Other Temperature Distributions

More realistic temperature distributions can be handled by superimposing solutions. Temperature distribution (3) approaches zero deep within the cavity. To overcome this difficulty, consider

$$T_w(x) = T_o + \Delta T\,e^{-mx} \tag{77}$$

$$T_w^4(x) = T_o^4 + 4T_o^3\Delta T\,e^{-mx} + 6T_o^2\Delta T^2\,e^{-2mx} + 4T_o\Delta T^3\,e^{-3mx} + \Delta T^4\,e^{-4mx} \tag{78}$$

The radiosity for this case is simply found by superposition:

$$B(x,m) = \varepsilon\sigma[T_o^4 B(x,0) + 4T_o^3\Delta T B(x,m) + 6T_o^2\Delta T^2 B(x,2m) + 4T_o\Delta T^3 B(x,3m) + \Delta T^4 B(x,4m)] + \rho\sigma T_b^4\phi(x) \tag{79}$$

likewise the over-all heat transfer is

$$Q^* = 2h\varepsilon\sigma[T_o^4 Q(0) + 4T_o^3\Delta T Q(m) + 6T_o^2\Delta T^2 Q(2m) + 4T_o\Delta T^3 Q(3m) + \Delta T^4 Q(4m) - T_b^4 Q_o] \tag{80}$$

Table 1 Radiosity at the edge of the cavity, $H(m)$

m	$\rho = 0.10$	$\rho = 0.50$	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.999$	$\rho = 0.9999$	$\rho = 1.00$
0.000	1.05409	1.41421	3.47850	4.47214	10.00000	34.7850	100.0000	∞
0.001	1.05404	1.41361	3.15467	4.45414	9.88551	30.2635	86.1421	522.220
0.002	1.05398	1.41302	3.14717	4.43648	9.77526	29.0463	76.0292	274.938
0.005	1.05381	1.41126	3.12519	4.38516	9.46564	26.0142	56.9397	118.963
0.010	1.05354	1.40839	3.09010	4.30441	9.00674	22.3216	40.9106	63.7895
0.020	1.05300	1.40285	3.02467	4.15748	8.24710	17.6404	26.9327	34.6355
0.030	1.05249	1.39755	2.96452	4.02625	7.63765	14.7521	20.4689	24.4162
0.040	1.05199	1.39246	2.90879	3.90773	7.13447	12.7738	16.7002	19.1331
0.050	1.05150	1.38755	2.85687	3.79983	6.71027	11.3259	14.2154	15.8807
0.060	1.05103	1.38280	2.80829	3.70097	6.34678	10.2162	12.4464	13.6663
0.080	1.05012	1.37376	2.71969	3.52565	5.75419	8.62017	10.0830	10.8284
0.100	1.04925	1.36525	2.64069	3.37444	5.28960	7.52160	8.56714	9.07591
0.150	1.04724	1.34591	2.47509	3.07200	4.46754	5.83834	6.39887	6.65388
0.200	1.04542	1.32882	2.34278	2.84330	3.92432	4.87391	5.23207	5.38909
0.250	1.04375	1.31353	2.23398	2.66314	3.53575	4.24351	4.49637	4.60464
0.300	1.04221	1.29973	2.14257	2.51695	3.24263	3.79681	3.98721	4.06738
0.400	1.03946	1.27569	1.99686	2.29298	2.82747	3.20217	3.32424	3.37452
0.500	1.03706	1.25538	1.88530	2.12857	2.54582	2.82176	2.90859	2.94387
0.600	1.03494	1.23792	1.79678	2.00218	2.34121	2.55608	2.62205	2.64860
0.800	1.03136	1.20936	1.66460	1.81979	2.06239	2.20746	2.25057	2.26769
1.000	1.02843	1.18688	1.57020	1.69391	1.88034	1.98765	2.01888	2.03112
1.500	1.02301	1.14706	1.42035	1.50109	1.61627	1.67903	1.69676	1.70367
2.000	1.01926	1.12090	1.33215	1.39136	1.47332	1.51669	1.52876	1.53343
2.500	1.01652	1.10240	1.27399	1.32044	1.38355	1.41634	1.42538	1.42887
3.000	1.01444	1.08864	1.23279	1.27088	1.32195	1.34817	1.35535	1.35812
4.000	1.01149	1.06963	1.17841	1.20626	1.24301	1.26159	1.26664	1.26858
5.000	1.00951	1.05715	1.14425	1.16611	1.19469	1.20902	1.21290	1.21438
6.000	1.00809	1.04837	1.12086	1.13883	1.16217	1.17380	1.17694	1.17814
8.000	1.00622	1.03690	1.09104	1.10425	1.12128	1.12971	1.13197	1.13284
10.000	1.00503	1.02976	1.07289	1.08332	1.09670	1.10330	1.10508	1.10575
15.000	1.00340	1.01999	1.04852	1.05534	1.06404	1.06831	1.06945	1.06989
20.000	1.00256	1.01503	1.03631	1.04137	1.04781	1.05097	1.05181	1.05213
25.000	1.00205	1.01203	1.02900	1.03302	1.03813	1.04063	1.04130	1.04155
30.000	1.00171	1.01003	1.02413	1.02747	1.03170	1.03377	1.03433	1.03454
40.000	1.00129	1.00752	1.01806	1.02055	1.02370	1.02525	1.02566	1.02581
50.000	1.00103	1.00602	1.01443	1.01641	1.01893	1.02015	1.02048	1.02061
100.000	1.000516	1.00301	1.00719	1.00818	1.00942	1.01003	1.01019	1.01026
150.000	1.000344	1.00200	1.00479	1.00545	1.00627	1.00668	1.00679	1.00683
200.000	1.000258	1.00150	1.00359	1.00408	1.00470	1.00500	1.00509	1.00512
500.000	1.000103	1.000601	1.00144	1.00163	1.00188	1.00200	1.00203	1.00204
1000.000	1.000052	1.000301	1.000718	1.000815	1.000939	1.00100	1.00102	1.00102

If ΔT is positive, the temperature distribution decreases with depth, while a negative ΔT means an increasing temperature distribution.

Almost any realistic emissive power distribution can be approximated by an exponential series

$$\sigma T_w^4(x) = \sum_{i=0}^{N} a_i e^{-b_i x} \tag{81}$$

where a_i and b_i are constants. Using superposition, the radiosity and over-all heat transfer can be expressed as

$$B(x) = \varepsilon \sum_{i=0}^{N} a_i B(x, b_i) + \rho\sigma T_b^4 \phi(x) \tag{82}$$

$$Q^* = 2h\varepsilon \sum_{i=0}^{N} a_i Q(b_i) - 2h\varepsilon\sigma T_b^4 Q_o \tag{83}$$

Results

The radiosity at the edge of the cavity is presented in Table 1 and Fig. 2 for a wide range of m and ρ values. These values were obtained by numerically solving integral Eq. (34) with the method of successive approximations. Longman's method[11] for computing infinite integrals with oscillatory integrands with Euler's transformation was utilized. For details the reader is referred to Ref. 10. The H-function decreases from $1/(\varepsilon)^{1/2}$ to unity as the temperature distribution parameter m is increased from zero to unity. When m is small, small changes in ρ from unity cause large changes in the H-function. When ρ is near unity a small variation from the isothermal case ($m = 0$) yields large changes in the H-function.

The dimensionless over-all heat transfer $Q(m)$ is calculated from Eq. (46) and is presented in Table 2 and Fig. 3. As expected, trends similar to those for the H-function are observed. The heat transfer is extremely sensitive to small changes in ρ when ρ is near unity. The dimensionless over-all heat transfer for the isothermal case ($m = 0$) is calculated from Eq. (48) and is tabulated in Table 2 and 3. Also, the coefficients for the large

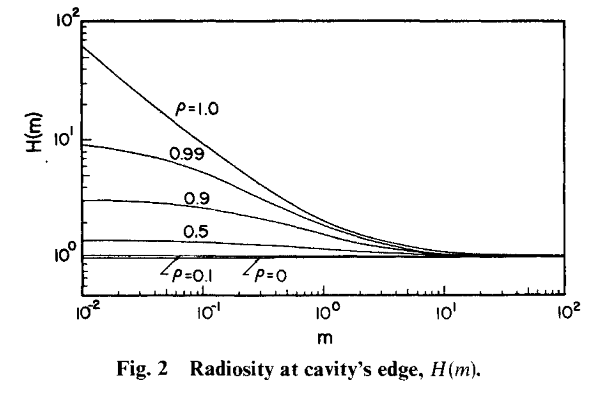

Fig. 2 Radiosity at cavity's edge, $H(m)$.

Table 2 Over-all heat transfer, $Q(m)$

m	$\rho = 0.10$	$\rho = 0.50$	$\rho = 0.90$	$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.999$	$\rho = 0.9999$	$\rho = 1.0$
0.000	0.542992	0.851981	2.69525	4.28174	11.8187	45.8279	166.673	∞
0.001	0.540887	0.848040	2.67323	4.23561	11.5646	43.0280	138.593	1000.000
0.002	0.539168	0.844794	2.65477	4.19675	11.3503	40.7788	119.866	500.000
0.005	0.534702	0.836319	2.60613	4.09438	10.7952	35.5073	86.1292	200.000
0.010	0.528333	0.824168	2.53611	3.94783	10.0330	29.4422	59.0953	100.000
0.020	0.517503	0.803428	2.41745	3.70307	8.85302	22.1302	36.5373	50.0000
0.030	0.508103	0.785404	2.31612	3.49837	7.95403	17.8010	26.5130	33.3333
0.040	0.499621	0.769155	2.22663	3.32120	7.23618	14.9164	20.8270	25.0000
0.050	0.491816	0.754227	2.14617	3.16490	6.64593	12.8497	17.1586	20.0000
0.060	0.484545	0.740352	2.07298	3.02524	6.15020	11.2935	14.5937	16.6667
0.080	0.471253	0.715092	1.94385	2.78473	5.36088	9.10168	11.2408	12.5000
0.100	0.459260	0.692440	1.83269	2.58371	4.75798	7.62909	9.14420	10.0000
0.150	0.433320	0.643995	1.60969	2.19706	3.72556	5.44128	6.24070	6.66667
0.200	0.411457	0.603845	1.43971	1.91694	3.06852	4.23360	4.73887	5.00000
0.250	0.392478	0.569556	1.30469	1.70317	2.61182	3.46670	3.82053	4.00000
0.300	0.375688	0.539690	1.19430	1.53401	2.27521	2.93605	3.20075	3.33333
0.400	0.347004	0.489742	1.02371	1.28230	1.81125	2.24910	2.41714	2.50000
0.500	0.323128	0.449247	0.897366	1.10324	1.50589	1.82336	1.94202	2.00000
0.600	0.302769	0.415516	0.799644	0.968944	1.28936	1.53348	1.62313	1.66667
0.800	0.269579	0.362136	0.657789	0.780371	1.00222	1.16391	1.22199	1.25000
1.000	0.243427	0.321495	0.559400	0.653926	0.820168	0.938083	0.979909	1.00000
1.500	0.196614	0.251872	0.408034	0.466208	0.564561	0.631902	0.655420	0.666667
2.000	0.165224	0.207406	0.321521	0.362570	0.430640	0.476496	0.492405	0.500000
2.500	0.142566	0.176390	0.265391	0.296733	0.348139	0.382467	0.394338	0.400000
3.000	0.125394	0.153475	0.225984	0.251166	0.292190	0.319442	0.328848	0.333333
4.000	0.101047	0.121830	0.174265	0.192177	0.221136	0.240267	0.246858	0.250000
5.000	0.0845963	0.100993	0.141813	0.155631	0.177885	0.192546	0.197594	0.200000
6.000	0.0727340	0.0862291	0.119547	0.130763	0.148785	0.160641	0.164722	0.166667
8.000	0.0567757	0.0667008	0.0909699	0.0990898	0.112105	0.120655	0.123597	0.125000
10.000	0.0465414	0.0543701	0.0734135	0.0797645	0.0899323	0.0966077	0.0989048	0.100000
15.000	0.0320607	0.0371674	0.0495132	0.0536154	0.0601748	0.0644790	0.0659603	0.0666667
20.000	0.024428	0.0282266	0.0373494	0.0403759	0.0452131	0.0483867	0.0494790	0.0500000
25.000	0.0197470	0.0227508	0.0299823	0.0323794	0.0362096	0.0387224	0.0395874	0.0400000
30.000	0.0165635	0.0190534	0.0250422	0.0270264	0.0301963	0.0322759	0.0329919	0.0333333
40.000	0.0125239	0.0143787	0.0188350	0.0203105	0.0226674	0.0242137	0.0247461	0.0250000
50.000	0.0100678	0.0115455	0.0150935	0.0162679	0.0181436	0.0193742	0.0197979	0.0200000
100.000	0.00508277	0.00581535	0.00757219	0.00815331	0.00908139	0.00969029	0.00989997	0.0100000
150.000	0.00339938	0.00388635	0.00505376	0.00543983	0.00605638	0.00646090	0.00660021	0.00666667
200.000	0.00255361	0.00291831	0.00379243	0.00408148	0.00454308	0.00484594	0.00495024	0.00500000
500.000	0.00102438	0.00116987	0.00151849	0.00163374	0.00181780	0.00193857	0.00198016	0.00200000
1000.000	0.000512678	0.000585361	0.000759495	0.000817064	0.000908996	0.000969315	0.000990089	0.001000

Table 3 Moments of the H-function

ρ	$\frac{1}{1+(\varepsilon)^{1/2}}$	$\frac{\alpha_1}{2(\varepsilon)^{1/2}}$	h_1	h_2	h_3	h_4
0	0.5000000	0.500000	0.0000000	0.00000000	0.000000	0.000000
0.1	0.513167	0.542992	0.0515504	0.00132872	−0.151395	−0.007801
0.2	0.527864	0.595072	0.106557	0.00567725	−0.305632	−0.032575
0.3	0.544467	0.659654	0.165682	0.0137253	−0.462767	−0.076754
0.4	0.563508	0.742186	0.229824	0.0264096	−0.622805	−0.143468
0.5	0.585786	0.851981	0.300266	0.0450798	−0.785651	−0.236900
0.6	0.612574	1.00654	0.378947	0.0718004	−0.950996	−0.362931
0.7	0.646111	1.24357	0.469074	0.110015	−1.11806	−0.530474
0.8	0.690983	1.66460	0.576739	0.166314	−1.28480	−0.754793
0.9	0.759747	2.69525	0.717027	0.257064	−1.44487	−1.06902
0.95	0.817256	4.28174	0.814633	0.331814	−1.51496	−1.28915
0.99	0.909091	11.8187	0.938178	0.440089	−1.54916	−1.55018
0.999	0.969347	45.8279	0.998340	0.498342	−1.53982	−1.66138
0.9999	0.990099	166.673	1.01441	0.514517	−1.53359	−1.68798
1.0000	1.000000	∞	1.02054	0.520755	−1.53063	−1.69758

m expansions are presented in Table 3. These coefficients were calculated using Longman's method.

Summary

Ambarzumian's method has been successfully applied to radiant interchange in a nonisothermal rectangular cavity. A exponentially decreasing temperature distribution was analyzed, and numerical results were presented for a wide range of temperature distributions m and reflectances ρ. As a result of this investigation, the following comments are made concerning Ambarzumian's method:

1) An integro-differential Eq. (23) for the radiosity distribution was developed.

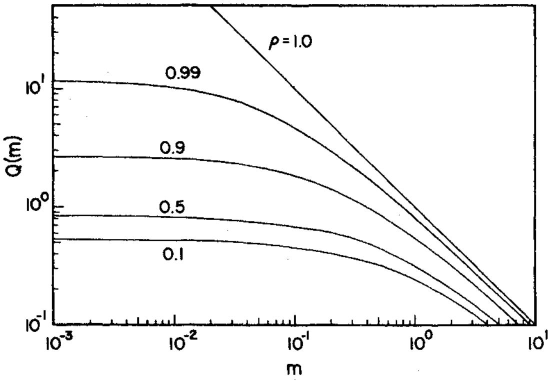

Fig. 3 Over-all heat transfer, $Q(m)$.

2) The radiosity at the edge of the cavity was shown to satisfy a nonlinear integral equation which was readily solved by iteration. For an isothermal cavity the radiosity at the edge is equal to $1/(\varepsilon)^{1/2}$.

3) The over-all heat transfer was related to the radiosity at cavity's edge. Thus, the over-all heat transfer was determined without calculating the entire radiosity distribution.

4) The radiosity $\phi(x)$ was simply related to radiosity for the isothermal case $B(0, m)$ by Eq. (44).

5) A large m expansion for the over-all heat transfer was found.

6) Taking into account the number and range of m values considered, the computational time was quite reasonable.

7) While the exponentially decreasing temperature distribution was essential to the method, solutions can be found to other temperature distributions which can be related to an exponential term.

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